Cooperative Games in Permutational Structure[1]

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Abstract

By a cooperative game in coalitional structure or shortly coalitional game we mean the standard cooperative non-transferable utility game described by a set of payoffs for each coalition being a nonempty subset of the grand coalition of all players. It is well-known that balancedness is a sufficient condition for the nonemptiness of the core of such a cooperative non-transferable utility game. In this paper we consider non-transferable utility games in which for any coalition the set of payoffs depends on a permutation or ordering upon any partition of the coalition into subcoalitions. We call such a game a cooperative game in permutational structure or shortly permutational game. Doing so we extend the scope of the standard cooperative game theory in dealing with economic or political problems. Next we define the concept of core for such games. By introducing balancedness for ordered partitions of coalitions, we prove the nonemptiness of the core of a balanced non-transferable utility permutational game. Moreover we show that the core of a permutational game coincides with the core of an induced game in coalitional structure, but that balancedness of the permutational game need not imply balancedness of the corresponding coalitional game. This leads to a weakening of the conditions for the existence of a nonempty core of a game in coalitional structure, induced by a game in permutational strucuture. Furthermore, we refine the concept of core for the class of permutational games. We call this refinement the balanced-core of the game and show that the balanced-core of a balanced permutational game is a nonempty subset of the core.

The proof of the nonemptiness of the core of a permututational game is based on a new intersection theorem on the unit simplex, which generalizes the well-known intersection theorem of Shapley.

Key words: non-transferable utility game, balancedness, core, unit simplex, closed covering, intersection theorem

1 Introduction

It is well-known that balancedness is a sufficient condition for the nonemptiness of the core of the standard cooperative non-transferable utility game described by a set of payoffs for each coalition being a nonempty subset of the grand coalition of all players. In the following we call such a non-transferable utility game a game in coalitional structure or shortly coalitional game. We also speak about coalitional balancedness if we mean the well-known concept of balancedness of a family of coalitions. Scarf [12] gave a constructive proof of the nonemptiness of the core of a coalitionally balanced game in coalitional structure based on the complementary pivoting technique introduced by Lemke and Howson [9]. Shapley [13] generalized the intersection theorem of Knaster-Kuratowski-Mazurkiewicz on the unit simplex in order to prove the nonemptiness of the core, see also Ichiishi [5]. In Billera [3] balancedness of a coalitional game has been generalized to π-balancedness of such a game.

In this paper we generalize the concept of a cooperative non-transferable utility game to a non-transferable utility game in which for any nonempty coalition a (possibly empty) set of attainable payoffs is given for any permutation or ordering upon a partition of the coalition into subcoalitions. This dependency on an ordered partition of the coalition reflects the situation in which the payoff set of a coalition is determined by the sequence in which the coalition is formed or on any hierarchy of the members of the coalition. In this way it is possible to differentiate between the players in the coalition, for instance between the player who takes the initiative to form the coalition, or is the most powerful player in the coalition, and the other players in the coalition. Another example is a situation when there is need for players to stand in a queue in order to get their payoff and waiting costs are involved. In such an environment it is necessary to differentiate the players in a coalition according to all the orderings of subsets of the coalition. Also for scheduling problems the outcome depends very much on the ordering of machines (i.e., players) to be processed.

Nowak and Radzik [11] have considered games in permutational form in case of transferable utilities and only permutations on the set of elements of a coalition are considered. For such TU games the value of the characteristic function depends on the ordering of the members of the coalition. Nowak and Radzik generalize the concept of the Shapley value for such games. We also refer to the work of Myerson [10], who used undirected graphs to model communication structures in cooperative games. In this paper we consider non-transferable utility games with payoff sets for any permutation upon each possible partition of the coalitions. We call such a game a non-transferable utility game in permutational structure or shortly permutational game. The core of a permutational game consists of all payoff vectors attainable for the grand coalition such that there is no coalition having a partition and permutation on the elements of this partition through which the coalition can improve upon the payoffs of all players in the coalition. Generalizing the concept of coalitional balancedness to balancedness for ordered partitions of coalitions, we prove the nonemptiness of the core of a balanced permutational game by applying a new intersection theorem on the unit simplex. Moreover we prove that the core of a permutational game coincides with the core of an induced game in coalitional structure. We also give an example showing that balancedness of the permutational game does not imply balancedness of the corresponding coalitional game. This therefore leads to a weakening of the conditions for the existence of a nonempty core of a game in coalitional structure. Next we refine the concept of core for a permutational game and show that for balanced permutational games this refinement is a nonempty subset of the core. We call this refinement the balanced-core of the game. By some examples we demonstrate that the balanced-core is indeed a proper subset of the core.

In Section 2 we introduce the concept of non-transferable utility permutational game. We also define for any permutational game an induced coalitional game and show that the core of the permutational game coincides with the core of the induced coalitional game. In Section 3 we define the concept of permutational balancedness and show that permutational balancedness of a permutational game does not imply coalitional balancedness of the induced coalitional game. In Section 4 we prove that balancedness of a permutational game is a sufficient condition for the nonemptiness of the core. This proof follows from a new intersection theorem on the unit simplex. If the induced coalitional game is not balanced, the nonemptiness of the core of this game follows from the nonemptiness of the core of the permutational game. In Section 5 we introduce the concept of balanced-core of a permutational game. In Section 6 we make some concluding remarks.

2 Permutational games

In an n-player cooperative non-transferable utility game introduced by Aumann and Peleg [2] each nonempty subset of players, called a coalition, can obtain any vector out of a certain subset of $\mathbb{R}^n$ as payoff vector. An attainable payoff vector lies in the core of the game if no coalition can improve upon this vector, see Aumann [1]. In this paper we introduce a cooperative non-transferable utility game in which the set of payoff vectors of a coalition is allowed to depend on the permutation or ordering on a partition of subcoalitions of the players in the coalition.

The set $\{1, \ldots, n\}$ of the n players in the game is denoted by $\mathcal{N}$. For a nonempty subset S of $\mathcal{N}$, called a coalition of players, let P^t_S denote a partition $\{S_1, \ldots, S_t\}$ of S into t subcoalitions of S and let $\pi(P^t_S) = (\pi_1(P^t_S), \ldots, \pi_t(P^t_S))$ denote a permutation or ordering of the elements of P_S^t. In the sequel a partition into t subcoalitions is called a t-partition and a permutation $\pi(P_S^t)$ on a t-partition of S is called an ordered t-partition of S. Furthermore, let Π_S^t be the set of all ordered t-partitions of S and let Π_S be the union over $t = 1, \ldots, s$ of all the sets Π_S^t, where $s = |S|$ denotes the number of elements of the set S. Finally, let $\mathcal{P}^{\mathcal{N}}$ denote the set of all ordered partitions of subsets of $\mathcal{N}$, i.e.,

$$\mathcal{P}^{\mathcal{N}} = \cup_{S \subset \mathcal{N}} \Pi_S.$$

Definition 2.1 Permutational Game
A non-transferable utility game in permutational structure or permutational game with n players is a function V from $\mathcal{P}^{\mathcal{N}}$ to the collection of subsets of $\mathbb{R}^n$ satisfying that for every $\pi(P_S^t) \in \mathcal{P}^{\mathcal{N}}$, the set $V(\pi(P_S^t)) \subset \mathbb{R}^n$ is a cylinder in the sense that for any two vectors u and v in $\mathbb{R}^n$ with $u_i = v_i$ for all $i \in S$, it holds that $u \in V(\pi(P_S^t))$ if and only if $v \in V(\pi(P_S^t))$.

In the sequel we denote a permutational game with n players and function V by the pair $(\mathcal{P}^{\mathcal{N}}, V)$. We call V the payoff function of the game $(\mathcal{P}^{\mathcal{N}}, V)$. If $u \in V(\pi(P_S^t))$ for some t-partition $\{S_1, \ldots, S_t\}$ of the coalition S, the members of S can guarantee themselves a payoff u_i for member $i \in S$, independent of what the players outside the coalition do, by agreeing on the permutation $\pi(P_S^t)$ of the t-partition $P_S^t = \{S_1, \ldots, S_t\}$ of S. In case S is the grand coalition $\mathcal{N}$, $V(\pi(P_{\mathcal{N}}^t))$ denotes the set of payoff vectors the players of the grand coalition can guarantee themselves when the players coordinate according to the permutation $\pi(P_{\mathcal{N}}^t)$. For ease of notation we define for any $S \subset \mathcal{N}$ the set of payoffs $V(S)$ by $V(S) = V(\pi(P_S^1))$, i.e., $V(S)$ is the set of payoff vectors the coalition S can guarantee itself without partitioning itself into subcoalitions.

For any permutational game $(\mathcal{P}^{\mathcal{N}}, V)$, let the function V' from the collection of subsets of $\mathcal{N}$ to the collection of subsets of $\mathbb{R}^n$ be defined by

$$V'(S) = \cup_{\pi \in \Pi_S} V(\pi),\ S \subset \mathcal{N}.$$

Then the function V' induces a non-transferable utility n-player game in coalitional structure, denoted by $(\mathcal{N}, V')$. Observe that $V(S) \subset V'(S)$, but that generally $V'(S)$ is not equal to $V(S)$. Moreover in Definition 2.1 we allow for empty payoff sets. Hence, it might be possible that some of the payoff sets $V'(S)$ are also empty. The core of the induced coalitional game $(\mathcal{N}, V')$, denoted by $C(\mathcal{N}, V')$, is as usual defined by the set of vectors $u \in V'(\mathcal{N})$ such that there do not exist a coalition $S \subset \mathcal{N}$ and a vector $v \in V'(S)$ such that $v_i > u_i$ for all $i \in S$. Analogously we say that a payoff vector u is in the core of the permutational game if $u \in V'(\mathcal{N})$ and there is no permutation $\pi(P_S^t)$ of a t-partition of a coalition S in which the coalition S can improve upon u.

Definition 2.2 Core of a Permutational Game
The core of a non-transferable utility permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ *is the set of vectors* $u \in \mathbb{R}^n$ *satisfying that* $u \in V'(\mathcal{N})$ *and there do not exist a coalition* S *with ordered partition* $\pi(P_S^t) \in \mathcal{P}^{\mathcal{N}}$ *and a vector* $v \in V(\pi(P_S^t))$ *such that* $v_i > u_i$ *for all* $i \in S$.

Observe that a core element is an element of $V'(\mathcal{N})$ because any vector u lying in a set $V(\pi(P_{\mathcal{N}}^t))$ of some permutation of some t-partition $P_{\mathcal{N}}^t$ of the grand coalition is attainable and hence the payoff set of the grand coalition is not restricted to the set $V(\mathcal{N})$. In the sequel we denote the core of a permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ by $C(\mathcal{P}^{\mathcal{N}}, V)$. Now we have the following lemma.

Lemma 2.3 Equivalence of Cores
For any permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ *and the induced coalitional game* $(\mathcal{N}, V')$ *it holds that* $C(\mathcal{P}^{\mathcal{N}}, V) = C(\mathcal{N}, V')$.

Proof.
For some $u \in \mathbb{R}^n$, first suppose $u \notin C(\mathcal{N}, V')$. Then there exists a coalition $S \subset \mathcal{N}$ and a vector $v \in \mathbb{R}^n$ such that $v \in V'(S)$ and $v_i > u_i$ for all $i \in S$. By the definition of $V'(S)$ this implies that there is some ordered partition $\pi(P_S^t)$ such that $v \in V(\pi(P_S^t))$. Hence $u \notin C(\mathcal{P}^{\mathcal{N}}, V)$. Secondly, suppose that $u \notin C(\mathcal{P}^{\mathcal{N}}, V)$. Then there exist an ordered partition $\pi(P_S^t)$ of some coalition S and some vector $v \in V(\pi(P_S^t))$ such that $v_i > u_i$ for all $i \in S$. By definition we have that $v \in V'(S)$. Hence $u \notin C(\mathcal{N}, V')$. $\square$

We conclude this section with an example of an economic situation which can be modelled as a permutational game.

Example 2.4
We consider a firm with n *employees. These employees have to perform all kinds of work, ranging from manual work (unskilled labour) to managerial work (high skilled labour). The employees have also different levels of skills. The problem is to which tasks which employees should be assigned. A coalition* S *denotes the set of employees getting a job, while the members outside* S *will be fired. Given a coalition* S*, the total amount work can be splitted up in* t *different tasks, with* t *ranging from* 1 *to* $|S|$*. For* $t = 1$*, each member of the coalition has to do the same task, including all types of work. So, in this case each member has to do for example both manual work and managerial work. For* $t = |S|$*, the work to be done is splitted up in specialized tasks as far as possible and all members of* S *have different tasks. Generally, for* $1 \leq t \leq |S|$ *we have* t *different tasks. To each task a group of employees of the set* S *will be assigned, yielding an ordered* t*-partition* P_S^t *of*

S. So, each member of S is assigned to precisely one task. We assume that these groups are ordered in such a way that the members of first group in this ordering are performing the most skilled labour and the members of the last group are performing the lowest skilled labour. The profit of the firm will depend on the ordered t-partition of subcoalitions and will be higher if the higher educated employees are asigned to the tasks needing more skills. Moreover it is assumed that the marginal utility of money for an employee in S depends on the task to which he is assigned. For instance, a high-skilled worker is more satisfied and therefore has a higher marginal utility of money when he is assigned to high-skilled tasks than when he is assigned to low-skilled tasks. For the permutation $\pi(P_S^t)$, let the number $b_i(\pi(P_S^t))$ denote the marginal utility of money for employee $i \in S$ and let its inverse be defined by $a_i(\pi(P_S^t)) = b_i^{-1}(\pi(P_S^t))$, $i \in S$. Furthermore, let $R(\pi(P_S^t))$ denote the profit if the tasks are divided according to $\pi(P_S^t)$. Finally, we assume that every employee has an outside option giving him payoff zero and that the firm can not be run by the employees if the assignment of the tasks is such that losses are made. Then the payoff sets are given by

$$V(\{i\}) = \{x \in \mathbb{R}^n \mid x_i \leq 0\},\ i \in \mathcal{N},$$

and for any ordered partition $\pi(P_{\mathcal{N}}^t)$ by

$$V(\pi(P_S^t)) = \emptyset \text{ if } R(\pi(P_S^t)) < 0.$$

and

$$V(\pi(P_S^t)) = \{x \in \mathbb{R}^n \mid \sum_{j \in S} a_j(\pi(P_S^t))x_j \leq R(\pi(P_S^t))\} \text{ if } R(\pi(P_S^t)) \geq 0.$$

For illustration, take $\mathcal{N} = \{1, 2\}$ and let the data be given as in the following table.

partition	R	a_1	a_2
$\{1,2\}$	*3*	$\frac{3}{2}$	*1*
$\{1\},\{2\}$	*4*	*4*	*1*
$\{2\},\{1\}$	*3*	*1*	*2*

The payoff sets corresponding to these data are drawn in Figure 1. The core of this game consists of all nonnegative elements on the boundary of $V'(\{1,2\})$, so $C(\mathcal{P}^{\mathcal{N}}, V) = \{x \in \mathbb{R}^2+ \mid x_2 = 4-4x_1 \text{ if } 0 \leq x_1 \leq \frac{2}{5},\ x_2 = 3-\frac{3x_1}{2} \text{ if } \frac{2}{5} \leq x_1 \leq \frac{3}{2},\ x_2 = 3/2-\frac{x_1}{2} \text{ if } \frac{3}{2} \leq x_1 \leq 3\}$.

3 Balanced permutational games

The core of a non-transferable utility permutational game might be empty. However, it will be shown that the core is nonempty if the permutational game satisfies some balancedness

Figure 1: Example 2.4, the payoff sets of the ordered partitions of $\{1, 2\}$

condition and every set $V(\pi(P_S^t))$, $\pi(P_S^t) \in \mathcal{P}^{\mathcal{N}}$, is comprehensive, closed and bounded from above in its projection space $\mathbb{R}^S$ defined by $\mathbb{R}^S = \{(x_i)_{i \in S} | x \in \mathbb{R}^n\}$. The balancedness condition differs from the well-known concept of balancedness of coalitions, in the sequel to be called coalitional balancedness. In this section we introduce the concept of permutational balancedness for ordered partitions of coalitions and define the related concept of a permutationally balanced game. Moreover we show by an example that permutational balancedness of the permutational game does not imply coalitional balancedness of the induced coalitional game. Since it will be proved in Section 4 that balancedness of the permutational game is sufficient for the nonemptiness of the core, it also follows that it is sufficient for the nonemptiness of the core of a coalitional game induced by a permutational game that the underlying permutational game is balanced.

First, for some coalition $S \subset \mathcal{N}$ and permutation $\pi(P_S^t)$ of a t-partition of S, we define the n-vector $m^{\pi(P_S^t)}$ by

$$m_j^{\pi(P_S^t)} = 0, \text{ if } j \notin S$$

and

$$m_j^{\pi(P_S^t)} = \frac{2(t-r+1)}{t(t+1)s_r}, \text{ if } j \in \pi_r(P_S^t),$$

where $s_r = |\pi_r(P_S^t)|$. Observe that $\sum_{j=1}^n m_j^{\pi(P_S^t)} = 1$. Furthermore, let m denote the vector all of whose components are equal to n^{-1}, i.e., $m = m^{\pi(P_{\mathcal{N}}^1)}$. For the ordered partition $\pi^j(P_{S^j}^{t_j}) = (\pi_1^j(P_{S^j}^{t_j}), \ldots, \pi_{t_j}^j(P_{S^j}^{t_j}))$, $j \in \{1, \ldots, k\}$, the vector $m^{\pi(P_S^t)}$ can be seen as the power vector of the members of coalition S in the ordered partition. Each member in a same subcoalition is assigned with the same power. The power of a subcoalition being the sum of the powers of its members depends on the rank of the subcoaliton in the ordering, in such a way that the power of a subcoalition $\pi_h^j(P_{S^j}^{t_j})$ is a fraction $\frac{t_j-h+1}{t_j}$ of the power of subcoalition $\pi_1^j(P_{S^j}^{t_j})$.

Example 3.1

Take $n = 3$ and consider the ordered 2-partition $\pi(P_{\{1,2\}}^2) = (\{1\}, \{2\})$ of the subset $\{1, 2\}$. Then $m^{\pi(P_S^t)} = (\frac{2}{3}, \frac{1}{3}, 0)^\top$. For the ordered 3-partition $\pi(P_{\mathcal{N}}^3) = (\{1\}, \{2\}, \{3\})$ we obtain $m^{\pi(P_{\mathcal{N}}^3)} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})^\top$. The ordered 2-partition $\pi(P_{\mathcal{N}}^2) = (\{1, 2\}, \{3\})$ of $\mathcal{N}$ gives $m^{\pi(P_{\mathcal{N}}^2)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^\top$ and the ordered 2-partition $\pi(P_{\mathcal{N}}^2) = (\{3\}, \{1, 2\})$ of $\mathcal{N}$ gives $m^{\pi(P_{\mathcal{N}}^2)} = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})^\top$. Observe that only the components $j \in S$ of the vector $m^{\pi(P_S^t)}$ get a positive power, the powers of two components is equal if they are in the same subset of the partition and that the total power of the components in some subset becomes greater if the subset has a higher priority in the ordering.

Definition 3.2 Permutational Balancedness
A family $\mathcal{B} = \{\pi^1, \ldots, \pi^k\}$ of k ordered partitions in $\mathcal{P}^{\mathcal{N}}$ is permutationally balanced if there exist positive numbers λ_j^*, $j = 1, \ldots, k$, such that

$$\sum_{j=1}^{k} \lambda_j^* m^{\pi^j} = m.$$

Permutational balancedness of a family $\mathcal{B}$ of k ordered partitions $\pi^1(P_{S^1}^{t_1}), \ldots, \pi^k(P_{S^k}^{t_k})$ in $\mathcal{P}^{\mathcal{N}}$ means that to any ordered partition $\pi^j(P_{S^j}^{t_j})$, $j = 1, \ldots, k$, a weight λ_j^* can be assigned in such a way that the total power of every player $i \in \mathcal{N}$ is the same and therefore equal to $m_i = \frac{1}{n}$. Geometrically it means that $\mathcal{B}$ is permutationally balanced if and only if the vector m lies in the relative interior of the convex hull of the vectors $m^{\pi^j(P_{S^j}^{t_j})}$, $j = 1, \ldots, k$. Notice that in Definition 3.2 it must hold that $\sum_{j=1}^{k} \lambda_j^* = 1$.

Example 3.3
Take $n = 3$. Then the family $\{\pi^1, \pi^2, \pi^3\}$ of two ordered 2-partitions and one 1-partition given by $\pi^1 = (\{1\}, \{2\})$, $\pi^2 = (\{2\}, \{3\})$ and $\pi^3 = (\{3\})$ is permutationally balanced. Since $m^{\pi^1} = (\frac{2}{3}, \frac{1}{3}, 0)^\top$, $m^{\pi^2} = (0, \frac{2}{3}, \frac{1}{3})^\top$, and $m^{\pi^3} = (0, 0, 1)^\top$, this family is permutational balanced with weights $\lambda_1^* = \frac{1}{2}$, $\lambda_2^* = \frac{1}{4}$, and $\lambda_3^* = \frac{1}{4}$. Observe that the ordered 2-partition $(\{1,2\}, \{3\})$ of $\mathcal{N}$ is permutationally balanced, but that the family of the ordered 2-partition $(\{1\}, \{2\})$ and the ordered 1-partition $(\{3\})$ is not permutationally balanced.

In case $\mathcal{B}$ is a family of 1-partitions we have that $\pi^j(P_{S^j}^{t_j}) = (S^j)$ and hence the system of equations in the balancedness condition reduces to

$$\sum_{j=1}^{k} \lambda_j^* m^{S^j} = m,$$

with $m_h^{S^j} = \frac{1}{|S^j|}$ if $h \in S^j$ and $m_h^{S^j} = 0$ if $h \notin S^j$, which is equal to the well-known concept of coalitional balancedness of the family of subsets $\{S^1, \ldots, S^k\}$ of $\mathcal{N}$. Therefore, the concept of permutational balancedness contains the concept of coalitional balancedness for a family of 1-partitions as a special case.

Definition 3.4 Balanced Permutational Game
A non-transferable utility permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ is permutationally balanced if for every permutationally balanced family $\mathcal{B} = \{\pi^1, \ldots, \pi^k\}$ of ordered partitions in $\mathcal{P}^{\mathcal{N}}$ it holds that

$$\cap_{i=1}^{k} V(\pi^i) \subset V'(\mathcal{N}).$$

In the sequel we speak shortly about a balanced permutational (or coalitional) game if we mean a permutationally (coalitionally) balanced non-transferable utility permutational (coalitional) game. For a given permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ any vector in the set $V'(S)$ is attainable for coalition S. Since $V(S) \subset V'(S)$, and generally $V'(S) \neq V(S)$, the induced coalitional game $(\mathcal{N}, V')$ need not to be coalitionally balanced if $(\mathcal{P}^{\mathcal{N}}, V)$ itself is permutationally balanced. This fact is shown in the next example.

Example 3.5
Take $n = 3$ and define the permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ by

$$V(i) = \{x \in \mathbb{R}^3 \mid x_i \leq 0\},\ i = 1, 2, 3,$$

$$V(1, 2) = \{x \in \mathbb{R}^3 \mid 2x_1 + x_2 \leq 3\},$$

and

$$V(2, 1) = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 \leq 3\},$$

where $V(i)$ denotes $V(\{i\})$ and $V(i, j)$ denotes $V((\{i\}, \{j\}))$. Furthermore,

$$V(\mathcal{N}) = V(3) \cap V(1, 2) \cap V(2, 1),$$

and

$$V(\pi) = \emptyset,\ \textit{otherwise}.$$

Observe again that we allow for empty payoff sets. The induced coalitional game is given by

$$V'(\{i\}) = V(i),\ i = 1, 2, 3,$$

$$V'(\{1, 2\}) = V(1, 2) \cup V(2, 1) \cup V(\{1, 2\}) = V(1, 2) \cup V(2, 1),$$

since $V(\{1, 2\}) = \emptyset$,

$$V'(\{1, 3\}) = V'(\{2, 3\}) = \emptyset,$$

and

$$V'(\mathcal{N}) = V(\mathcal{N}).$$

The projection of the sets $V(1, 2)$ and $V(2, 1)$ on the (x_1, x_2)-space is given in Figure 2. The shaded area in this figure is the projection of the set $V(\{1, 2, 3\}) = V'(\{1, 2, 3\})$ on the (x_1, x_2)-space. Both the permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ and the coalitional game $(\mathcal{N}, V')$ have the point $(1, 1, 0)^\top$ as the unique core element. For the permutational game this point lies *in $V(\mathcal{N})$ and there is no coalition having an ordered partition through which the coalition can improve upon this outcome. The coalition $\{1,2\}$ can improve on each other point in $V(\mathcal{N})$ through the ordered 2-partition $(\{1\},\{2\})$ or the ordered 2-partition $(\{2\},\{1\})$. Also for the coalitional game the outcome $(1,1,0)^\top$ is the unique element of $V'(\mathcal{N})$ on which the coalition $\{1,2\}$ cannot improve upon. Clearly the coalitional game is not balanced, since the family of coalitions $\{1,2\}$ and $\{3\}$ is coalitionally balanced, whereas the point $x = (\frac{1}{2},2,0)^\top$ lies in $V'(\{1,2\}) \cap V'(\{3\})$ but not in $V'(\mathcal{N})$. On the other hand the permutational game is permutationally balanced. In fact there are only four relevant families of ordered partitions to consider, namely the family of the three ordered 1-partitions $(\{1\})$, $(\{2\})$, $(\{3\})$, the family of two ordered 2-partitions and one ordered 1-partition $(\{1\},\{2\})$, $(\{2\},\{1\})$, $(\{3\})$, the family of one ordered 2-partition and two ordered 1-partitions $(\{1\},\{2\})$, $(\{2\})$, $(\{3\})$, and the family of one ordered 2-partition and two ordered 1-partitions $(\{2\},\{1\})$, $(\{1\})$, $(\{3\})$. For each of these families we have that the intersection of the sets of payoffs of the members of the family is a subset of $V'(\mathcal{N})$, for instance $V(1,2) \cap V(2) \cap V(3) \subset V'(\mathcal{N})$. For all other permutationally balanced families we have that the intersection of the payoff sets of the members of the family is empty and hence is a subset of $V'(\mathcal{N})$.*

4 Nonemptiness of the core of a balanced permutational game

In order to prove the nonemptiness of the core of a balanced permutational game we first introduce an intersection theorem on the $(n-1)$-dimensional unit simplex Δ defined by

$$\Delta = \{x \in \mathbb{R}^n + \mid \sum_{j=1}^{n} x_j = 1\}.$$

In this theorem the simplex Δ is covered by closed subsets C^π, $\pi \in \mathcal{P}^{\mathcal{N}}$, satisfying some boundary condition. Under this condition there exists a balanced collection of permutations for which the corresponding subsets of Δ have a nonempty intersection. This is stated in the next lemma, which is a generalization of the well-known intersection theorem of Shapley [13], in which only sets C^S are defined for coalitions $S \subset \mathcal{N}$. The lemma is also closely related to intersection theorems given in Ichiishi and Idzik [6] and in van der Laan, Talman and Yang [8].

Lemma 4.1
Let $\{C^{\pi(P^t_S)} \mid \pi(P^t_S) \in \mathcal{P}^{\mathcal{N}}\}$ be a collection of closed sets covering Δ satisfying that if x lies in the boundary of Δ and $x \in C^{\pi(P^t_S)}$, then $S \subset \{i \in \mathcal{N} \mid x_i > 0\}$. Then there is

Figure 2: Example 3.4, balanced permutational game

a permutationally balanced family $\{\pi^1, \ldots, \pi^k\}$ *of* k *ordered partitions in* $\mathcal{P}^{\mathcal{N}}$ *for which it holds that*

$$\cap_{j=1}^{k} C^{\pi^j} \neq \emptyset.$$

Proof.

For any ordered partition $\pi \in \mathcal{P}^{\mathcal{N}}$, define the vector $c^{\pi} = m - m^{\pi}$. For $x \in \Delta$, define the set $F(x)$ by

$$F(x) = \mathrm{Conv}(\{c^{\pi} \mid x \in C^{\pi}\}),$$

where $\mathrm{Conv}(X)$ denotes the convex hull of a set $X \subset \mathbb{R}^n$. Clearly, for every $x \in \Delta$, the set $F(x)$ is nonempty, convex, and compact. Moreover, $\cup_{x \in \Delta} F(x)$ is bounded and F is an upper hemi-continuous mapping from the set Δ to the collection of subsets of the set Y^n defined by

$$Y^n = \{y \in \mathbb{R}^n \mid m^{\top} y = 0 \text{ and } y_i \geq -1 \text{ for } i = 1, \ldots, n\}.$$

Both sets Δ and Y^n are nonempty, convex, and compact. Next, let G be the mapping from Y^n to the collection of subsets of Δ defined by

$$G(y) = \{x \in \Delta \mid x'^{\top} y \leq x^{\top} y \text{ for every } x' \in \Delta\}.$$

Clearly, for every $y \in Y^n$ the set $G(y)$ is nonempty, convex, and compact, and G is upper hemi-continuous. Hence, the mapping H from the nonempty, convex, compact set $\Delta \times Y^n$ into the collection of subsets of $\Delta \times Y^n$ defined by $H(x, y) = G(y) \times F(x)$ is upper hemi-continuous and for every $(x, y) \in \Delta \times Y^n$, the set $H(x, y)$ is nonempty, convex, and compact. According to Kakutani's fixed point theorem the mapping H has a fixed point on $\Delta \times Y^n$, i.e., there exist $x^* \in \Delta$ and $y^* \in Y^n$ satisfying $y^* \in F(x^*)$ and $x^* \in G(y^*)$. Let $\alpha^* = x^{*\top} y^*$. From $x^* \in G(y^*)$ it follows that $x^{\top} y^* \leq \alpha^*$ for every $x \in \Delta$. By taking $x = e(i)$, where $e(i)$ is the i-th unit vector, we obtain that $y_i^* \leq \alpha^*$, $i = 1, \ldots, n$. Hence, $y_i^* = \alpha^*$ if $x_i^* > 0$ and $y_i^* \leq \alpha^*$ if $x_i^* = 0$. On the other hand, $y^* \in F(x^*)$ implies there exist nonnegative numbers $\lambda_1^*, \ldots, \lambda_k^*$ satisfying $\sum_{j=1}^{k} \lambda_j^* = 1$ and $y^* = \sum_{j=1}^{k} \lambda_j^* c^{\pi^j}$, where π^j, $j = 1, \ldots, k$, are such that $x^* \in C^{\pi^j}$. Without loss of generality we may assume that $\lambda_j^* > 0$ for every $j = 1, \ldots, k$. We now show that $y^* = \underline{0}$ and hence that the collection $\{\pi^1, \ldots, \pi^k\}$ is permutationally balanced. Since by definition of the set Y^n, $m^{\top} y^* = 0$, we obtain that $\alpha^* \geq 0$. Moreover, by the boundary condition we have that $x_i^* = 0$ implies that $i \notin S^j$ for every $j = 1, \ldots, k$, with S^j the set satisfying $\pi^j = \pi^j(P_{S^j}^{tj})$, and hence $c_i^{\pi^j} = \frac{1}{n}$. Hence, when $x_i^* = 0$, then $y_i^* = \sum_{j=1}^{k} \lambda_j^* c_i^{\pi^j} = \sum_{j=1}^{k} \lambda_j^* n^{-1} > 0$. Therefore, $0 < y_i^* \leq \alpha^*$ if $x_i^* = 0$ and $y_i^* = \alpha^* \geq 0$ if $x_i^* > 0$. Since $\sum_{j=1}^{n} y_i^* = 0$, this implies that $x_i^* > 0$ for every $i \in \mathcal{N}$ and $\alpha^* = 0$. So, $y^* = \underline{0}$. Consequently, $\{\pi^1, \ldots, \pi^k\}$ is permutationally balanced.

Since $x^* \in \cap_{j=1}^k C^{\pi^j}$, this completes the proof. $\square$

By applying Lemma 4.1 we can prove the nonemptiness of the core of a balanced permutational game.

Theorem 4.2
A non-transferable utility permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ has a nonempty core if
i) the set $V(\{i\})$ is given by $V(\{i\}) = \{x \in \mathbb{R}^n \mid x_i \leq \alpha_i\}$ for some $\alpha_i \in \mathbb{R}$,
ii) the game is permutationally balanced,
iii) for every $S \subset \mathcal{N}$ and for every $\pi \in \Pi_S$, the set $V(\pi)$ is comprehensive and closed, and the set $\{(x_i)_{i \in S} \in \mathbb{R}^S \mid x \in V(\pi)$ and $x_i \geq \alpha_i$ for all $i \in S\}$ is bounded.

Proof.
Without loss of generality we may assume that $\underline{0} \in V(\{i\})$ for any $i \in \mathcal{N}$. To prove the theorem we define a closed covering $\{C^\pi \mid \pi \in \mathcal{P}^{\mathcal{N}}\}$ of Δ satisfying the conditions of Lemma 4.1 and show that an intersection point of a permutationally balanced collection of these sets induces an element in the core of the game. For given $M > 0$ and for any $x \in \Delta$, let the number λ_x be determined by

$$\lambda_x = \max\{\lambda \in \mathbb{R} \mid -Mx + \lambda m \in \cup_{\pi \in \mathcal{P}^{\mathcal{N}}} V(\pi)\}.$$

Since $\underline{0} \in V(\{i\})$ and because of iii), for every $M > 0$, λ_x exists for any $x \in \Delta$. Moreover, following Shapley [13], by the condition of boundedness from above, $M > 0$ can be chosen so large that for every $i \in \mathcal{N}$ and $x \in \Delta$, $x_i = 0$ implies that $i \notin S$ for any S satisfying $-Mx + \lambda_x m \in V(\pi(P_S^t))$. Now, for $\pi \in \mathcal{P}^{\mathcal{N}}$ we define

$$C^\pi = \{x \in \Delta \mid -Mx + \lambda_x m \in V(\pi)\}.$$

Since every $V(\pi)$ is closed and comprehensive, the collection of sets $\{C^\pi | \pi \in \mathcal{P}^{\mathcal{N}}\}$ is a collection of closed sets covering the simplex Δ, and satisfying the boundary condition of Lemma 4.1. Hence there is a balanced family $\mathcal{B} = \{\pi^1, \ldots, \pi^k\}$ of elements of $\mathcal{P}^{\mathcal{N}}$ such that $\cap_{j=1}^k C^{\pi^j} \neq \emptyset$. Let x^* be a point in this intersection, so $x^* \in C^{\pi^j}$ for $j = 1, \ldots, k$. Since the game is balanced we have that $\cap_{j=1}^k V(\pi^j) \subset V'(\mathcal{N})$ and hence $u^* = -Mx^* + \lambda_{x^*} m \in V'(\mathcal{N})$. Now, suppose there exist a vector $v \in \mathbb{R}^n$ and an ordered partition $\pi(P_S^t) \in \mathcal{P}^{\mathcal{N}}$ of a coalition S such that $v \in V(\pi(P_S^t))$ and $v_i > u_i^*$ for all $i \in S$. Since $V(\pi(P_S^t))$ is comprehensive and cylindric, there is a $\mu > 0$ such that $u^* + \mu m \in V(\pi(P_S^t))$. However, then $-Mx^* + (\lambda_{x^*} + \mu) m \in V(\pi(P_S^t))$, which contradicts that $-Mx^* + \lambda m \notin V(\pi(P_S^t))$ for any $\lambda > \lambda_{x^*}$. Hence $u^* \in C(\mathcal{P}^{\mathcal{N}}, V)$. $\square$

5 Balanced-core of permutational games

For permutational games the concept of the core can be refined to what we will call the balanced-core. The balanced-core consists of all elements of the core that can be supported by a balanced collection of ordered partitions of coalitions and will be denoted by $BC(\mathcal{P}^{\mathcal{N}}, V)$. We show that the balanced-core is nonempty if the game is permutationally balanced. In case the permutational game happens to be a coalitional game its balanced-core is equal to the core. In general the balanced-core of a permutational game is a proper subset of the core of this game, and so the balanced-core of a permutationally balanced game is a nonempty subset of its core.

Definition 5.1 Balanced-core
The balanced-core of a non-transferable utility permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ is the set of all vectors $u \in V'(\mathcal{N})$ satisfying that
i) for any $S \subset \mathcal{N}$, there do not exist an ordered partition $\pi(P_S^t) \in \mathcal{P}^{\mathcal{N}}$ and a vector $v \in V(\pi(P_S^t))$ such that $v_i > u_i$ for all $i \in S$,
ii) there exists a permutationally balanced family $\{\pi^1, \ldots, \pi^k\}$ of k ordered partitions in $\mathcal{P}^{\mathcal{N}}$, such that $u \in \cap_{j=1}^k V(\pi^j))$.

Clearly, it follows immediately from condition i) of Definition 5.1 that a payoff vector in the balanced-core lies also in the core of the permutational game. Condition ii) says that an element of the core is an elememt of the balanced-core only if it lies in the intersection of the payoff sets of a balanced collection of ordered partitioned coalitions. One could say that a core element lies in the balanced-core if it is supported by a balanced family of ordered partitions of coalitions. Since every player participates with equal weight in a balanced family of coalitions this gives some appealing stability property to the elements in the balanced-core. All players have an equal weight in supporting a balanced-core payoff vector. In some economic situations only one ordered partitioning of a coalition may actually be formed. In such situations the weight of an ordered partitioned coalition in the permutationally balanced family of ordered partitioned coalitions supporting the balanced-core element selected as the outcome can be interpreted as the probability with which the ordered partitioned coalition indeed forms. The following theorem is straightforward.

Theorem 5.2
A non-transferable utility permutational game $(\mathcal{P}^{\mathcal{N}}, V)$ satisfying the conditions i)-iii) of Theorem 4.2 has a nonempty balanced-core.

Proof.
In the proof of Theorem 4.2 it is shown that there exists a core element u^* satisfying

$$u^* \in \cap_{j=1}^k V(\pi^j)$$

for a permutationally balanced collection $\{\pi^1, \ldots, \pi^k\}$ of elements of $\mathcal{P}^{\mathcal{N}}$. Clearly, such an element u^* is in $BC(\mathcal{P}^{\mathcal{N}}, V)$. $\square$

Observe that a permutational game is essentially a coalitional game if for any ordered partitioning $\pi(P_S^t)$ with $t > 1$ it holds that $V(\pi(P_S^t)) \subset V(S)$, implying that for the induced coalitional game it holds that $V'(S) = V(S)$. In this case the balanced-core is equal to the core, because also $V'(\mathcal{N}) = V(\mathcal{N})$. Hence, in this case any core element lies in $V(\mathcal{N})$ and therefore is stable with respect to the permutationally balanced family $\{\mathcal{N}\}$. However, in general the set $V(\mathcal{N})$ is a proper subset of $V'(\mathcal{N})$. In this case the core may contain many elements that do not belong to the balanced-core. This is illustrated in the next examples.

Example 5.3
(i) Consider the permutational game defined in Example 2.4 for $n = 2$. *We have seen already that the core consists of all nonnegative elements on the boundary of* $V'(\mathcal{N})$. *However, the balanced-core is given by* $BC(\mathcal{P}^{\mathcal{N}}, V) = \{x \in \mathbb{R}^2+ \mid x_2 = 3 - \frac{3x_1}{2},\ \frac{2}{5} \leq x_1 \leq \frac{3}{2}\}$, *with the unpartitioned coalition* $\{1,2\}$ *as the unique element of the supporting permutationally balanced family of coalitions. By giving the players equal votes the unpartitioned coalition* $\{1,2\}$ *forms and the payoff is divided according to some balanced-core element.*

(ii) Consider the same example, except that we take $R(\pi(P_{\mathcal{N}}^1)) = 1$, *i.e.,*

$$V(\{1,2\}) = \{x \in \mathbb{R}^2 \mid 3x_1 + 2x_2 \leq 1\}.$$

Then again the core consists of all nonnegative elements on the boundary of $V'(\mathcal{N})$ *and is given by* $C(\mathcal{P}^{\mathcal{N}}, V) = \{x \in \mathbb{R}^2+ \mid x_2 = 4 - 4x_1$ *if* $0 \leq x_1 \leq \frac{5}{7}$, $x_2 = 3/2 - \frac{x_1}{2}$ *if* $\frac{5}{7} \leq x_1 \leq 3\}$. *However, the balanced-core is given by* $BC(\mathcal{P}^{\mathcal{N}}, V) = \{(\frac{5}{7},\ \frac{8}{7})^\top\}$ *and its unique element is supported by the permutationally balanced family* $\{(\{1\},\{2\}), (\{2\},\{1\})$ *with a weight of* $\frac{1}{2}$ *for both members of this family. Both partitioned coalitions may form with probability* $\frac{1}{2}$.

(iii) Finally, let the payoff sets be defined by $V(\{i\}) = \{x \in \mathbb{R}^2 \mid x_i \leq 0\}$, $i = 1,2$, $V(\{1\},\{2\}) = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 2\}$, *and* $V(\{2\},\{1\}) = V(\{1,2\}) = \emptyset$. *Then* $C(\mathcal{P}^{\mathcal{N}}, V) = \{x \in \mathbb{R}^2+ \mid x_1 + x_2 = 2\}$ *and* $BC(\mathcal{P}^{\mathcal{N}}, V) = \{(2,0)^\top\}$. *The unique element of the balanced-core is supported by the permutationally balanced family* $\{(\{1\},\{2\}), (\{2\})\}$ *with a weight of* $3/4$ *for the ordered partition* $(\{1\},\{2\})$ *of coalition* $\{1,2\}$ *and a weight of* $1/4$ *for the one person coalition* $\{2\}$.

6 Concluding remarks

In this paper we introduced permutational games and proved that the (balanced-)core of such a game is nonempty if the game is permutationally balanced. This concept of balancedness is a generalization of the well-known concept of balancedness of coalitions. Analogously the existence result concerning the nonemptiness of the core is more general than for games in coalitional structure. A game in coalitional structure is a special case in the family of games in permutational structure. Indeed, when $V(\pi(P_S^t)) = \emptyset$ for every $t \geq 2$, then the permutational game is a game in coalitional structure and permutational balancedness coincides with coalitional balancedness. In general the induced coalitional game $(\mathcal{N}, V')$ of a balanced permutational game, need not to be coalitionally balanced. Since a permutational game and its induced coalitional game have the same core, it follows that permutational balancedness of the underlying permutational game is a sufficient condition for the nonemptiness of the core.

Billera [3] has pointed out that in case of coalitional games there are many ways to define the powers of the players. In the same way there is a lot of freedom to define the powers in case of permutational games. For example, for a given ordered partition $\pi(P_S^t) \in \mathcal{P}^{\mathcal{N}}$ of a coalition S, one could define the n-dimensional power vector $m^{\pi(P_S^t)}$ by

$$m_j^{\pi(P_S^t)} = 0, \text{ if } j \notin S$$

and

$$m_j^{\pi(P_S^t)} = \frac{t^{r-1}(t+1)^{1-r}}{\sum_{h=1}^{t} s_h t^{h-1}(t+1)^{1-h}}, \text{ if } j \in \pi_r(P_S^t),$$

where $s_h = |\pi_h(P_S^t)|$. It is easily seen that $\sum_{j=1}^{n} m_j^{\pi(P_S^t)} = 1$. In this case we have that $m_k^{\pi(P_S^t)} > m_l^{\pi(P_S^t)}$ for any $k \in \pi_i(P_S^t)$ and $l \in \pi_j(P_S^t)$ if $1 \leq i < j \leq t$. This implies that every member in a higher ranked subcoalition has always more power than any member in a lower ranked subcoalition. Notice however that this has consequences in forming permutationally balanced families and hence on the fact whether or not a game in permutational structure is permutationally balanced. Since the core of a game does not depend on the definition of the power vectors, this implies that for the nonemptiness of the core of a permutational game it is sufficient to have permutational balancedness with respect to some collection of power vectors. Notice that if we take $m^{\pi(P_S^t)} = m^S$ for every $\pi(P_S^t) \in \mathcal{P}^{\mathcal{N}}$, then the permutational game is balanced with respect to these constant (for every S) vectors if and only if the induced coalitional game is balanced. Hence, an induced coalitionally game being balanced implies that the original permutational game is permutationally balanced with respect to some collection of power vectors. Clearly, the other way around is not true, i.e., a coalitional game induced by a balanced permutational game may not be coalitionally balanced with respect to any set of power vectors. While the core is independent of the choice of the power vectors, the balanced core does depend on this choice. The choice of the power vectors should depend on the economic situation under consideration. The appropriate power vectors in a given application is a point of further research.

In Kamiya and Talman [7] a simplicial algorithm was proposed to find a core element of a coalitional game. Similarly, we can apply the simplicial algorithm on the unit simplex Δ of Doup and Talman [4] to find approximately an element of the balanced-core and so a core element of a balanced permutational game.

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